Statistical Inference
Test Set 1

1. Let $X \sim \mathrm{P}(\lambda)$. Find unbiased estimators of $(i)\ \lambda^3$, $(ii)\ e^{-\lambda}\cos\lambda$, (iii) $\sin\lambda$. (iv) Show that there does not exist unbiased estimators of $1/\lambda$, and $\exp\{-1/\lambda\}$.
2. Let $X_1, X_2, ..., X_n$ be a random sample from a $N(\mu, \sigma^2)$ population. Find unbiased and consistent estimators of the signal to noise ration $\frac{\mu}{\sigma}$ and quantile $\mu + b\sigma$, where b is any given real.
3. Let $X_1, X_2, ..., X_n$ be a random sample from a $U(-\theta, 2\theta)$ population. Find an unbiased and consistent estimator of θ.
4. Let X_1, X_2 be a random sample from an exponential population with mean $1/\lambda$. Let $T_1 = \frac{X_1 + X_2}{2}, T_2 = \sqrt{X_1 X_2}$. Show that T_1 is unbiased and T_2 is biased. Further, prove that $MSE(T_2) \le Var(T_1)$.
5. Let T_1 and T_2 be unbiased estimators of θ with respective variances σ_1^2 and σ_2^2 and $\mathrm{cov}(T_1, T_2) = \sigma_{12}$ (assumed to be known). Consider $T = \alpha T_1 + (1-\alpha)T_2,\ 0 \le \alpha \le 1.$ Show that T is unbiased and find value of α for which $Var(T)$ is minimized.
6. Let $X_1, X_2, ..., X_n$ be a random sample from an $Exp(\mu, \sigma)$ population. Find the method of moment estimators (MMEs) of μ and σ.
7. Let $X_1, X_2, ..., X_n$ be a random sample from a Pareto population with density $f_X(x) = \frac{\beta\alpha^\beta}{x^{\beta+1}},\ x > \alpha,\ \alpha > 0,\ \beta > 2.$ Find the method of moments estimators of $\alpha, \beta.$
8. Let $X_1, X_2, ..., X_n$ be a random sample from a $U(-\theta, \theta)$ population. Find the MME of θ.
9. Let $X_1, X_2, ..., X_n$ be a random sample from a lognormal population with density $f_X(x) = \frac{1}{\sigma x\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(\log_e x - \mu)^2\right\},\ x > 0.$ Find the MMEs of μ and σ^2.
10. Let $X_1, X_2, ..., X_n$ be a random sample from a double exponential (μ, σ) population. Find the MMEs of μ and σ.

Hints and Solutions

1. (i) $E\{X(X-1)(X-2)\} = \lambda^3$

(ii) For this we solve estimating equation. Let $T(X)$ be unbiased for $e^{-\lambda}\cos\lambda$. Then

$$ET(X) = e^{-\lambda}\cos\lambda \text{ for all } \lambda>0 .$$

$$\Rightarrow \sum_{x=0}^{\infty} T(x)\frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\cos\lambda \text{ for all } \lambda>0$$

$$\Rightarrow \sum_{x=0}^{\infty} T(x)\frac{\lambda^x}{x!} = 1-\frac{\lambda^2}{2!}+\frac{\lambda^4}{4!}-\cdots\cdots \text{ for all } \lambda>0$$

As the two power series are identical on an open interval, equating coefficients of powers of λ on both sides gives

$$\begin{aligned} T(x) &= 0, \text{ if } x = 2m+1, \\ &= 1, \text{ if } x = 4m, \\ &= -1, \text{ if } x = 4m+2,\ m = 0,1,2,\ldots \end{aligned}$$

(iii) For this we have to solve estimating equation. However, we use Euler's identity to solve it.

Let $U(X)$ be unbiased for $\sin\lambda$. Then

$$\Rightarrow \sum_{x=0}^{\infty} U(x)\frac{\lambda^x}{x!} = \frac{1}{2i}e^{\lambda}(e^{i\lambda}-e^{-i\lambda}) \text{ for all } \lambda>0$$

$$= \frac{1}{2i}(e^{(1+i)\lambda}-e^{(1-i)\lambda}) \text{ for all } \lambda>0$$

$$= \frac{1}{2i}\left(\sum_{k=0}^{\infty}\frac{\lambda^k(1+i)^k}{k!} - \sum_{k=0}^{\infty}\frac{\lambda^k(1-i)^k}{k!}\right) \text{for all } \lambda>0.$$

Applying De-Moivre's Theorem on the two terms inside the parentheses, we get

$$\sum_{x=0}^{\infty} U(x)\frac{\lambda^x}{x!} = \frac{1}{2i}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\left[(\sqrt{2})^k\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^k-(\sqrt{2})^k\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)^k\right]$$

$$= \frac{1}{2i}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\left[(\sqrt{2})^k\left(\cos\frac{k\pi}{4}+i\sin\frac{k\pi}{4}\right)-(\sqrt{2})^k\left(\cos\left(-\frac{k\pi}{4}\right)+i\sin\left(-\frac{k\pi}{4}\right)\right)\right]$$

$$= \sum_{k=0}^{\infty}\frac{(\sqrt{2})^k\lambda^k}{k!}\sin\left(\frac{k\pi}{4}\right) \text{ for all } \lambda>0$$

Equating the coefficients of powers of λ on both sides gives

$$U(x) = (\sqrt{2})^x\sin\left(\frac{\pi x}{4}\right),\ x = 0,1,2,\ldots$$

In Parts (iv) and (v) , we can show in a similar way that estimating equations do not have any solutions.

2. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$.

Then $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$, and W=$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$. It can be seen that

$E(W^{1/2}) = \frac{\sqrt{2}\,\Gamma\frac{n}{2}}{\Gamma\frac{n-1}{2}}$ and $E(W^{-1/2}) = \frac{\Gamma\frac{n-2}{2}}{\sqrt{2}\,\Gamma\frac{n-1}{2}}$. Using these, we get unbiased estimators of σ and $\frac{1}{\sigma}$ as $T_1 = \sqrt{\frac{n-1}{2}}\frac{\Gamma\frac{n-1}{2}}{\Gamma\frac{n}{2}} S$ and $T_2 = \sqrt{\frac{2}{n-1}}\frac{\Gamma\frac{n-1}{2}}{\Gamma\frac{n-2}{2}}\frac{1}{S}$ respectively. As $\bar{X}$ and S^2 are independently distributed, $U_1 = \bar{X}\,T_2$ is unbiased for $\frac{\mu}{\sigma}$. Further, $U_2 = \bar{X} + bT_1$ is unbiased for $\mu + b\sigma$. As $\bar{X}$ and S^2 are consistent for μ and σ^2 respectively, U_1 and U_2 are also consistent for $\frac{\mu}{\sigma}$ and $\mu + b\sigma$ respectively.

3. As $\mu_1' = \frac{3\theta}{2}, T = \frac{2\bar{X}}{3}$ is unbiased for θ. T is also consistent for θ.

4. As $E(X_i) = \frac{1}{\lambda}, T_1$ is unbiased. Also X_1 and X_2 are independent. So

$E(T_2) = E\left(\sqrt{X_1X_2}\right) = \left(E(\sqrt{X_1})\right)^2 = \left(\frac{1}{2}\sqrt{\frac{\pi}{\lambda}}\right)^2 = \frac{\pi}{4\lambda}$. $Var(T_1) = \frac{1}{2\lambda^2}$.

$$MSE(T_2) = E\left(\sqrt{X_1X_2} - \frac{1}{\lambda}\right)^2 = E(X_1X_2) - \frac{2}{\lambda}E\left(\sqrt{X_1X_2}\right) + \frac{1}{\lambda^2}$$

$$= \frac{2}{\lambda^2}\left(1 - \frac{\pi}{4}\right)$$

5. The minimizing choice of α is obtained as $\frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$.

6. $f(x) = \frac{1}{\sigma}\exp\left(-\frac{x-\mu}{\sigma}\right), x > \mu, \sigma > 0.$ $\mu_1' = \mu + \sigma, \mu_2' = (\mu + \sigma)^2 + \sigma^2$.

So $\mu = \mu_1' - \sqrt{\mu_2' - \mu_1'^2}, \sigma = \sqrt{\mu_2' - \mu_1'^2}$. The method of moments estimators for μ and σ are therefore given by

$\hat{\mu}_{MM} = \bar{X} - \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2}$, and $\hat{\sigma}_{MM} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2}$.

7. $\mu_1' = \dfrac{\beta\alpha}{\beta - 1}$, $\mu_2' = \dfrac{\beta\alpha^2}{\beta - 2}$. So $\alpha = \dfrac{\mu_1'\sqrt{\mu_2'}}{\sqrt{\mu_2' - \mu_1'^2}}$, $\beta = 1 + \sqrt{\dfrac{\mu_2'}{\mu_2' - \mu_1'^2}}$

The method of moments estimators for α and β are therefore given by

$$\hat{\alpha}_{MM} = \frac{\bar{X}\sqrt{\sum_{i=1}^{n} X_i^2}}{\sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2} + \sqrt{\sum_{i=1}^{n} X_i^2}},\ \hat{\beta}_{MM} = 1 + \sqrt{\frac{\sum_{i=1}^{n} X_i^2}{\sum_{i=1}^{n}(X_i - \bar{X})^2}}.$$

8. Since $\mu_1' = 0$, we consider $\mu_2' = \dfrac{\theta^2}{3}$. So $\hat{\theta}_{MM} = \sqrt{\dfrac{3}{n}\sum_{i=1}^{n} X_i^2}$

9. $\mu_1' = e^{\mu + \sigma^2/2}$, $\mu_2' = e^{2\mu + 2\sigma^2}$. So $\mu = \log\left(\dfrac{\mu_1'^2}{\sqrt{\mu_2'}}\right)$, $\sigma^2 = \log\left(\dfrac{\mu_2'}{\mu_1'^2}\right)$ and the method of moments estimators for μ and σ^2 are therefore given by

$$\hat{\mu}_{MM} = \log\left(\frac{\bar{X}^2}{\sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2}}\right),\ \hat{\sigma}^2_{MM} = \log\left(\frac{\frac{1}{n}\sum_{i=1}^{n} X_i^2}{\bar{X}^2}\right).$$

10. $f(x) = \dfrac{1}{2\sigma}\exp\left(-\left|\dfrac{x-\mu}{\sigma}\right|\right), x \in \mathbb{R}, \mu \in \mathbb{R}, \sigma > 0.$ $\mu_1' = \mu, \mu_2' = \mu^2 + 2\sigma^2$.

So $\mu = \mu_1'$, $\sigma = \sqrt{\dfrac{1}{2}(\mu_2' - \mu'^2)}$. The method of moments estimators for μ and σ are therefore given by

$\hat{\mu}_{MM} = \bar{X}$, and $\hat{\sigma}_{MM} = \sqrt{\dfrac{1}{2n}\sum_{i=1}^{n}(X_i - \bar{X})^2}$.